Published by Institute of Physics Publishing for SISSA/ISAS
Received: *January 07, 2002*
Revised: *May 07, 2002*
Accepted: *June 09, 2002*

Towards an explicit expression of the Seiberg-Witten map at all orders

Stéphane Fidanza

*Centre de Physique Théorique, École Polytechnique,**
91128 Palaiseau Cedex, France
E-mail: fidanza@cpht.polytechnique.fr

Abstract: The Seiberg-Witten map links non-commutative gauge theories to ordinary gauge theories, and allows to express the non-commutative variables in terms of the commutative ones. Its explicit form can be found order by order in the non-commutative parameter θ and the gauge potential A by the requirement that gauge orbits are mapped on gauge orbits. This of course leaves ambiguities, corresponding to gauge transformations, and there is an infinity of solutions. Is there one better, clearer than the others? In the abelian case, we were able to find a solution, linked by a gauge transformation to already known formulas, which has the property of admitting a recursive formulation, uncovering some pattern in the map. In the special case of a pure gauge, both abelian and non abelian, these expressions can be summed up, and the transformation is expressed using the parametrisation in terms of the gauge group.

Keywords: Duality in Gauge Field Theories, Brane Dynamics in Gauge Theories.

*Unité mixte du CNRS et de l'EP, UMR 7644

JHEP06(2002)016

Contents

1. Introduction

Noncommutativity has been studied extensively since it has become clear that non-commutative gauge theories can describe the low energy effective action of a brane in a constant magnetic background (see [1, 2] for reviews). In particular in [3], a correspondence, known as the Seiberg-Witten map, between gauge fields living on D-brane worldvolume in the background of a non-vanishing constant electromagnetic field and non-commutative gauge theory on a space with coordinate x^μ satisfying

$$[x^\mu; x^\nu] = i\theta^{\mu\nu} \tag{1.1}$$

has been established.

The simplest example of noncommutative gauge theory is the abelian U(1), which is not a free theory, and is more similar to a U(N) gauge theory. In fact, at least on a torus, there exists a transformation, the Morita equivalence, that allows to change the non-commutativity parameter of the space and the rank of the gauge group at the same time.

JHEP06(2002)016

For example on a 2-torus, U(1) with non-commutativity parameter $\theta = 1/N$ is equivalent to an ordinary U(N) theory. In this context, Morita equivalence can be seen from rewriting the objects in a matrix language [4, 5].

The action of Seiberg-Witten (SW) map and Morita equivalence could be combined, for example, to link the usual abelian Born-Infeld action to its non-abelian version [6], for which the ambiguities have not yet been solved [7, 8, 9, 15]. In this process, Morita equivalence brings all these ordering ambiguities back into the non-commutative world, where we know that the DBI lagrangian is the lagrangian invariant under SW map, in the slowly varying fields limit. Fixing the ambiguities requires going beyond this limit, which means also having a better knowledge and understanding of the SW map.

Another related issue is about the dynamics of the B field. For instance, if we take two snapshots of a U(1) theory, with two different B fields, they would correspond to two different non-commutativity parameters. Let us say that the first one is linked to an ordinary U(5) gauge group by Morita equivalence, while the second one is related to U(3). Would that mean that the rank of the gauge group itself has become dynamic? We believe that there are still things to be understood in the case of a constant θ, that could help dealing with a dynamic background.

In this paper, we would like to explore further the order by order formulation of the Seiberg-Witten map, and the structures associated with it. There already exists a formula, known as Liu's conjecture [10], recently proven in [11]–[14], which expresses the commutative field strength for a U(1) gauge group in terms of the non-commutative variables non perturbatively. This formula was found by considering the couplings of the non-commutative brane to Ramond-Ramond potentials, and its order by order expansion involves the $\star_n$ products [16, 17].

Our approach here is completely different. It is based on solving order by order the Seiberg-Witten equation, and is closer in spirit to [19]. Using the freedom in the possible solutions (see [18]) we have also chosen expressions that differ from those of [17] by a gauge transformation. The hope was to express the map in a simpler way, by extending the recursive formulation that appears in the usual first orders of the field strength (see 3.1.5). Our guiding criterion for choosing a solution is to preserve its structure at all orders. Hence, our expressions, if still order by order, can be expressed recursively and explicitly written at any order with little work, at least in the abelian case. Hopefully, these recursive expressions will lead to the full non-perturbative formula, as in the case of the pure gauge, where we were able to express the solution in a particularly suggestive form. Of course, what is interesting in the pure gauge case is not finding a solution but rather a clue on the particular form of the general solution.

The structure of the paper is as follows: definitions and notations will be found in section 2, as well as some remarks about our method. Section 3 contains the results for an abelian gauge theory, and the order by order solutions are expressed recursively. This is the main perturbative feature of the paper, and has been explicitly checked up to order 6 (cf. 3.1). Work still remains to be done to extend the pattern to all orders in A_μ (cf. 3.2). A non-perturbative explanation for the existence of this structure has been found only in the case of the pure gauge. Then indeed, the solution can be compactly expressed using the parametrisation in terms of the gauge group, be it abelian (cf. 3.3) or not (cf. 4.1). The same kind of parametrisation may be possible in the general case, but has not yet been achieved. Some remarks about solving the non-abelian equations are made in 4.2.

2. Discovering the landscape

Let us consider a commutative gauge theory. The gauge group may be general, and will not be explicit. The gauge potential is A_μ, with field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i\,[A_\mu ; A_\nu]\,.$$

The gauge transformation δ, with gauge parameter λ, will be acting as

$$\delta A_\mu = \partial_\mu \lambda + i\,[\lambda; A_\mu] = D_\mu \lambda\,, \qquad \delta F_{\mu\nu} = i\,[\lambda; F_{\mu\nu}]\,.$$

We will first study the abelian theory, so that all commutators vanish and the expressions simplify. When working with a non-abelian theory, the gauge structure will then be encoded in the commutators and anticommutators.

On the non-commutative side, usual multiplication of functions is replaced by star product, which here is the Moyal-Weyl formula for a constant non-commutativity parameter $\theta^{\mu\nu}$:

$$\begin{aligned} f \star g &= f\, e^{\frac{i}{2}\overleftarrow{\partial_\rho}\theta^{\rho\sigma}\overrightarrow{\partial_\sigma}}\, g \\ &= fg + \frac{i}{2}\,(\partial_\rho f)\,\theta^{\rho\sigma}\,(\partial_\sigma g) - \frac{1}{8}\,(\partial_\rho\partial_\kappa f)\,\theta^{\rho\sigma}\theta^{\kappa\tau}\,(\partial_\sigma\partial_\tau g) + O(\theta^3)\,. \end{aligned} \tag{2.1}$$

This allows us to write the non-trivial star commutator[1]

$$i[f \overset{\star}{;} g] = i[f;g] - \frac{1}{2}\{\partial f \overset{\theta}{;} \partial g\} - \frac{i}{8}[\partial\partial f \overset{\theta\theta}{;} \partial\partial g] + \frac{1}{48}\{\partial\partial\partial f \overset{\theta\theta\theta}{;} \partial\partial\partial g\} + \cdots \tag{2.2}$$

which, in the abelian case, reduces to

$$i[f \overset{\star}{;} g] = -\partial f\,\theta\,\partial g + \frac{1}{24}\,\partial\partial\partial f\,\theta\theta\theta\,\partial\partial\partial g + \cdots\,. \tag{2.3}$$

In particular, we have the noncommutativity of the coordinates: $[x^\mu \overset{\star}{;} x^\nu] = i\theta^{\mu\nu}$.

On this space lives the non-commutative gauge theory with gauge potential $\hat{A}_\mu$ and field strength

$$\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - i[\hat{A}_\mu \overset{\star}{;} \hat{A}_\nu]\,.$$

The gauge transformation $\hat{\delta}$ has gauge parameter $\hat{\lambda}$ and

$$\hat{\delta}\hat{A}_\mu = \partial_\mu \hat{\lambda} + i[\hat{\lambda} \overset{\star}{;} \hat{A}_\mu] = \hat{D}_\mu \hat{\lambda}\,, \qquad \hat{\delta}\hat{F}_{\mu\nu} = i[\hat{\lambda} \overset{\star}{;} \hat{F}_{\mu\nu}]\,.$$

[1] where we have introduced the notations $\{\partial f \overset{\theta}{;} \partial g\} = \theta^{ij}\{\partial_i f; \partial_j g\}$,

$$[\partial\partial f \overset{\theta\theta}{;} \partial\partial g] = \theta^{ij}\theta^{kl}[\partial_i\partial_k f; \partial_j\partial_l g], \quad \{\partial\partial\partial f \overset{\theta\theta\theta}{;} \partial\partial\partial g\} = \theta^{ij}\theta^{kl}\theta^{mn}\{\partial_i\partial_k\partial_m f; \partial_j\partial_l\partial_n g\}\,.$$

The first operand carries the first index of θ, making these symbols antisymmetric. Generally speaking, we will not write the indices when contractions are done in a natural way.

2.1 The Seiberg-Witten map

The Seiberg-Witten map is a map between the commutative and the non-commutative gauge theory, which is compatible with gauge transformations. In other words, it maps gauge orbits into gauge orbits.

We can write non-commutative objects as functions of the commutative ones

$$\hat{A}_\mu = \hat{A}_\mu(A_\mu;\theta)\,; \qquad \hat{F}_{\mu\nu} = \hat{F}_{\mu\nu}(A_\mu;\theta)\,; \qquad \hat{\lambda} = \hat{\lambda}(\lambda, A_\mu;\theta)$$

and we have the following diagram, which tells that gauge transforming $\hat{A}_\mu$ as the non-commutative gauge potential or as a function of the commutative one is the same:

$$\begin{array}{ccc} A_\mu & \xrightarrow{\;\delta\;} & A_\mu + \delta A_\mu \\ \downarrow & & \downarrow \\ \hat{A}_\mu & \xrightarrow{\;\hat{\delta}\;} & \hat{A}_\mu + \hat{\delta}\hat{A}_\mu \end{array} \iff \widehat{A_\mu + \delta A_\mu} = \hat{A}_\mu + \hat{\delta}\hat{A}_\mu \iff \delta\hat{A}_\mu = \hat{\delta}\hat{A}_\mu\,.$$

This is the Seiberg-Witten equation, which will allow us to find the map explicitly:

$$\delta\hat{A}_\mu - \partial_\mu\hat{\lambda} = i[\hat{\lambda} \stackrel{\star}{,} \hat{A}_\mu]\,. \tag{2.4}$$

2.2 Freedom in the solution

To solve eq. (2.4), we develop the non-commutative variables in powers of θ:

$$\begin{aligned} \hat{A}_\mu &= A_\mu + A_\mu^{(1)} + A_\mu^{(2)} + \cdots\,, \\ \hat{F}_{\mu\nu} &= F_{\mu\nu} + F_{\mu\nu}^{(1)} + F_{\mu\nu}^{(2)} + \cdots\,, \\ \hat{\lambda} &= \lambda + \lambda^{(1)} + \lambda^{(2)} + \cdots\,. \end{aligned}$$

Then the left hand side of eq. (2.4) is, at order n, exactly $\delta A_\mu^{(n)} - \partial_\mu\lambda^{(n)}$, while development of the star commutator on the right hand side only involves terms of lower (or same) order. This is the key why we can find explicit solution to this equation, order by order in θ. Of course, since the only requirement was mapping gauge orbits into gauge orbits, there is some freedom in the solution. In fact, any "homogeneous solution", meaning$(\mathbb{A}_\mu^{(n)};\Lambda^{(n)})$ so that $\delta\mathbb{A}_\mu^{(n)} - \partial_\mu\Lambda^{(n)} = 0$, can be freely added to a particular solution of the complete equation. This "homogeneous solution" does not change the physics, and can be rewritten as a gauge transformation [18]. We will see an example of it in section 3.1.2, where we relate our solution with the already known solutions of [10, 12].

In the following, there will be another type of development: order by order in powers of A (figure 1). In fact, each term in a solution can only involve, algebraically, A, θ and derivatives. Specifying the number of A and the number of θ then characterizes a class of terms (freedom remains in the contraction of indices, and the action of derivatives). We will denote a term of order $(A^n;\theta^m)$ in the development of $\hat{\lambda}$ or $\hat{A}_\mu$ by $\lambda^{(n,m)}$ or $A_\mu^{(n,m)}$.

Let us stress that this order in A, whereas well defined in the abelian theory, becomes rather unclear in a non-abelian theory. The field strength, the covariant derivative and the gauge transformation contain a quadratic part in A. This does not mean that order in A

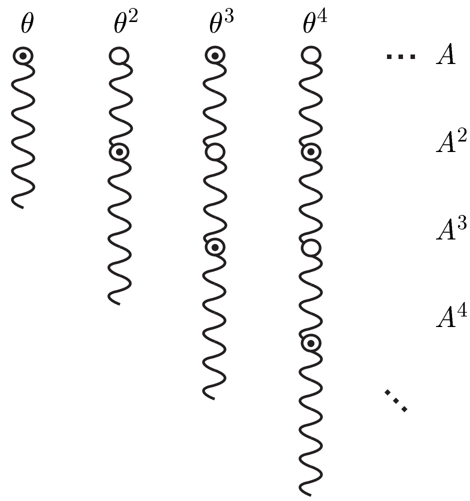

Figure 1: Circles are non-abelian terms, that spread on higher orders in A, hence the tails. When the group is abelian, many terms drop out, leaving only the dots, which do not spread anymore.

does not make sense, but that we will have to be very careful in the identification of the terms. Misidentification would spoil the picture.

We will also see that it may be more efficient to sum all the terms of a given order in A, and not in θ, as advertised. Thus, we will use the notation $\lambda^{(n,\infty)}$, $A_\mu^{(n,\infty)}$ meaning order n in A

$$\lambda^{(n,\infty)} = \sum_{m=0}^{\infty} \lambda^{(n,m)}\,; \qquad A_\mu^{(n,\infty)} = \sum_{m=0}^{\infty} A_\mu^{(n,m)}\,.$$

3. The abelian theory

In the abelian theory, different orders in A do not mix together. We will start with the first diagonal of the diagram: the terms of order (A^n, θ^n). We will see that we can write solutions so that each point is given from the previous one. The expressions can then also be written easily without recursivity. The emphasized expressions have been checked up to and including $n = 6$. For $n = 4, 5, 6$, this has been done using a computer program specially written on this purpose.

3.1 The first diagonal: orders (A^n, θ^n)

3.1.1 First order

The first order equation is reduced to $\delta A_\mu^{(1,1)} - \partial_\mu \lambda^{(1,1)} = -\partial\lambda\theta\partial A_\mu$ and admits the usual solution [3]

$$\begin{aligned}
\lambda^{(1,1)} &= -\frac{1}{2}\,\theta^{\rho\sigma} A_\rho \partial_\sigma \lambda\,,\\
A_\mu^{(1,1)} &= -\frac{1}{2}\,\theta^{\rho\sigma} A_\rho (\partial_\sigma A_\mu + F_{\sigma\mu})\,.
\end{aligned}$$

We can rewrite this solution to make its structure more apparent, so that we can recognize the form of next orders

$$\begin{aligned} \lambda^{(1,1)} &= -\frac{1}{2}\,(A\theta\partial)\lambda\,, \\ A_\mu^{(1,1)} &= -\frac{1}{2}\,(A\theta\partial)A_\mu - \frac{1}{2}\,(A\theta F)_\mu\,. \end{aligned} \tag{3.1}$$

3.1.2 Second order

The equation at order (A^2,θ^2) is $\delta A_\mu^{(2,2)} - \partial_\mu\lambda^{(2,2)} = -\partial\lambda^{(1,1)}\theta\partial A_\mu - \partial\lambda\theta\partial A_\mu^{(1,1)}$.

Solving with the previous (1,1) expressions, we get the solution

$$\begin{aligned} \lambda^{(2,2)} &= \frac{1}{6}[(A\theta\partial)(A\theta\partial)\lambda + A\theta F\theta\partial\lambda]\,, \\ A_\mu^{(2,2)} &= \frac{1}{6}[(A\theta\partial)(A\theta\partial)A_\mu + A\theta F\theta\partial A_\mu + 2(A\theta\partial)(A\theta F)_\mu + 2A\theta F\theta F_\mu]\,. \end{aligned} \tag{3.2}$$

But there are other possibilities. In [19] another solution was found

$$\begin{aligned} \lambda^{(2,2)}{}_{\mathrm{JMSSW}} &= \frac{1}{2}\,\theta^{\rho\sigma}\theta^{\kappa\tau}A_\rho\partial_\sigma A_\kappa\partial_\tau\lambda\,, \\ A_\mu^{(2,2)}{}_{\mathrm{JMSSW}} &= \frac{1}{2}\,\theta^{\rho\sigma}\theta^{\kappa\tau}A_\rho[\partial_\sigma A_\kappa\partial_\tau A_\mu + A_\kappa\partial_\sigma F_{\tau\mu} + F_{\sigma\kappa}F_{\tau\mu}]\,. \end{aligned}$$

The difference between the two is the "homogeneous solution" $1/6H$, where H is

$$\begin{aligned} \Lambda^{(2,2)} &= \theta^{\rho\sigma}\theta^{\kappa\tau}A_\rho[\partial_\kappa A_\sigma + \partial_\sigma A_\kappa - A_\kappa\partial_\sigma]\partial_\tau\lambda = \delta[A\theta(\partial A)\theta A]\,, \\ \mathbb{A}_\mu^{(2,2)} &= \theta^{\rho\sigma}\theta^{\kappa\tau}A_\rho[\partial_\kappa A_\sigma + \partial_\sigma A_\kappa - A_\kappa\partial_\sigma]\partial_\mu A_\tau = \partial_\mu[A\theta(\partial A)\theta A]\,. \end{aligned}$$

In particular, it is a gauge transformation on $\hat{A}_\mu$, with parameter $(1/6)A\theta(\partial A)\theta A$.

The solution (3.2) is also different from the order 2 expansion of Liu's formula [10, 16, 17]. This time, the difference is $(5/12)H$, and it is again a gauge transformation. Of course, in both cases, the field strength is the same.

3.1.3 Third order

The third order begins to get very messy. The equation is

$$\delta A_\mu^{(3,3)} - \partial_\mu\lambda^{(3,3)} = -\partial\lambda^{(2,2)}\theta\partial A_\mu - \partial\lambda^{(1,1)}\theta\partial A_\mu^{(1,1)} - \partial\lambda\theta\partial A_\mu^{(2,2)}\,.$$

Expanding it with the actual expressions (3.1) and (3.2) gives a great number of terms. But we can build an expression on the model arising from (3.1) and (3.2). And indeed, we explicitly checked that the following is a solution to this equation

$$\begin{aligned} \lambda^{(3,3)} &= -\frac{1}{24}\,[\,(A\theta\partial)(A\theta\partial)(A\theta\partial)\lambda + (A\theta F\theta\partial)(A\theta\partial)\lambda + \\ &\qquad + 2(A\theta\partial)(A\theta F\theta\partial)\lambda + 2A\theta F\theta F\theta\partial\lambda\,]\,, \\ A_\mu^{(3,3)} &= -\frac{1}{24}\,[\,(A\theta\partial)(A\theta\partial)(A\theta\partial)A_\mu + (A\theta F\theta\partial)(A\theta\partial)A_\mu + \\ &\qquad + 2(A\theta\partial)(A\theta F\theta\partial)A_\mu + 2A\theta F\theta F\theta\partial A_\mu + \\ &\qquad + 3(A\theta\partial)(A\theta\partial)(A\theta F)_\mu + 3(A\theta F\theta\partial)(A\theta F)_\mu + \\ &\qquad + 6(A\theta\partial)(A\theta F\theta F)_\mu + 6A\theta F\theta F\theta F_\mu\,]\,. \end{aligned} \tag{3.3}$$

The interest of having this explicit solution is that we see a structure emerging. The operators look very similar, the coefficients seem to follow some rule. The first two orders were not really enough to draw conclusions, but order 3 strongly suggests that the form can be generalized to a solution of order n. Furthermore, we developed a computer code which allowed us to check that these expressions are also solutions at order 4, 5 and 6.

3.1.4 Recursive formulation

Better than describing some prescriptions leading to the general form of order n – or rather (n,n), we can express the results obtained so far in a recursive manner.

Recalling the first order solution

$$\begin{aligned} \lambda^{(1,1)} &= -\frac{1}{2}\, A\theta\partial\lambda\,, \\ A_\mu^{(1,1)} &= -\frac{1}{2}\, A\theta(\partial A_\mu + F_{\cdot\mu}) \end{aligned}$$

and comparing to (3.2) and (3.3), we can write the second and third orders in a similar way, with $A^{(1,1)}$ or $A^{(2,2)}$ *acting as operators*, under the form given above. That means explicit derivatives (contrary to those hidden in Fs) also act on what is on the right.

$$\begin{aligned} \lambda^{(2,2)} &= -\frac{1}{3}\, A^{(1,1)}\theta\partial\lambda\,, & \lambda^{(3,3)} &= -\frac{1}{4}\, A^{(2,2)}\theta\partial\lambda\,, \\ A_\mu^{(2,2)} &= -\frac{1}{3}\, A^{(1,1)}\theta(\partial A_\mu + 2F_{\cdot\mu})\,, & A_\mu^{(3,3)} &= -\frac{1}{4}\, A^{(2,2)}\theta(\partial A_\mu + 3F_{\cdot\mu})\,. \end{aligned}$$

This leads to the conjecture that the same iterative relation gives a solution for general n

$$\begin{aligned} \lambda^{(n,n)} &= -\frac{1}{n+1}\, A^{(n-1,n-1)}\theta\partial\lambda\,, \\ A_\mu^{(n,n)} &= -\frac{1}{n+1}\, A^{(n-1,n-1)}\theta(\partial A_\mu + nF_{\cdot\mu})\,. \end{aligned} \tag{3.4}$$

It is also possible to express an algebraically factorized form without recursivity. Let us denote as d and f the two basic blocks that are building all the terms

$$d \equiv A\theta\partial\,, \qquad f \equiv A\theta F A^{-1}$$

with $A_\rho^{-1}A_\sigma = \delta_{\rho\sigma}$ and the rule $A^{-1}\lambda = 0$ (or $f\lambda = 0$). Then

$$\begin{aligned} \lambda^{(n,n)} &= \frac{(-1)^n}{(n+1)!}\,(d+f)(d+2f)\cdots(d+nf)\,\lambda\,, \\ A_\mu^{(n,n)} &= \frac{(-1)^n}{(n+1)!}\,(d+f)(d+2f)\cdots(d+nf)\,A_\mu\,. \end{aligned}$$

The only effect of the $f\lambda = 0$ rule is that when developing the last factor, the formulas for A have twice the number of terms than those for λ, as has been seen previously. It might be interesting to learn more about the significance and properties of these f and d, even if they can be thought of as the first order of more general objects. Indeed, we will see the first correction in section 3.2.

3.1.5 The field strength

We calculated the non-commutative field strength corresponding to the previous expressions for $\hat{A}_\mu$

$$
\begin{aligned}
F^{(1,1)}_{\mu\nu} &= \partial_\mu A^{(1,1)}_\nu - \partial_\nu A^{(1,1)}_\mu + \partial A_\mu \theta \partial A_\nu \;=\; -(A\theta\partial)F_{\mu\nu} - F_{\mu\cdot}\theta F_{\cdot\nu}\,,\\
F^{(2,2)}_{\mu\nu} &= \partial_\mu A^{(2,2)}_\nu - \partial_\nu A^{(2,2)}_\mu + \partial A^{(1,1)}_\mu \theta \partial A_\nu + \partial A_\mu \theta \partial A^{(1,1)}_\nu\\
&= \frac{1}{2}(A\theta\partial)(A\theta\partial)F_{\mu\nu} + \frac{1}{2}(A\theta F\theta\partial)F_{\mu\nu} + (A\theta\partial)(F_{\mu\cdot}\theta F_{\cdot\nu}) + F_{\mu\cdot}\theta F\theta F_{\cdot\nu}\,,\\
F^{(3,3)}_{\mu\nu} &= \partial_\mu A^{(3,3)}_\nu - \partial_\nu A^{(3,3)}_\mu + \partial A^{(2,2)}_\mu \theta \partial A_\nu + \partial A_\mu \theta \partial A^{(2,2)}_\nu + \partial A^{(1,1)}_\mu \theta \partial A^{(1,1)}_\nu\\
&= -\frac{1}{6}\left[(A\theta\partial)(A\theta\partial)(A\theta\partial)F_{\mu\nu} + 2(A\theta\partial)(A\theta F\theta\partial)F_{\mu\nu} +\right.\\
&\qquad \left. +\; (A\theta F\theta\partial)(A\theta\partial)F_{\mu\nu} + 2(A\theta F\theta F\theta\partial)F_{\mu\nu}\right] -\\
&\quad - \frac{1}{2}\left[(A\theta\partial)(A\theta\partial)(F_{\mu\cdot}\theta F_{\cdot\nu}) + (A\theta F\theta\partial)(F_{\mu\cdot}\theta F_{\cdot\nu})\right] -\\
&\quad - \left[(A\theta\partial)(F_{\mu\cdot}\theta F\theta F_{\cdot\nu}) + F_{\mu\cdot}\theta F\theta F\theta F_{\cdot\nu}\right].
\end{aligned}
$$

And again we identify in these expressions the same recursive pattern

$$
\begin{aligned}
F^{(1,1)}_{\mu\nu} &= -(A\theta\partial)F_{\mu\nu} - F_{\mu\cdot}\theta F_{\cdot\nu}\,,\\
F^{(2,2)}_{\mu\nu} &= -(A^{(1,1)}\theta\partial)F_{\mu\nu} - F^{(1,1)}_{\mu\cdot}\theta F_{\cdot\nu}\,,\\
F^{(3,3)}_{\mu\nu} &= -(A^{(2,2)}\theta\partial)F_{\mu\nu} - F^{(2,2)}_{\mu\cdot}\theta F_{\cdot\nu}\,.
\end{aligned}
\tag{3.5}
$$

Let us stress that the second line only involves usual formulas: $F^{(1,1)}_{\mu\nu}$, $F^{(2,2)}_{\mu\nu}$ and $A^{(1,1)}_\mu$ are the same here as in [10, 19]. The particular choice that we have made for $A^{(2,2)}_\mu$ and others then allowed the same pattern for the third line.

3.2 A line by line relation?

Now that we have some expression for the first diagonal, it still remains to fill the entire triangle on figure 1 to get all the terms. In fact, the whole first line (order 1 in A) is really easy to find, and the solution is essentially unique. It is known as a $\star_2$ or $\star'$ product expression [16]:

$$
A^{(1,\infty)}_\mu = -\frac{1}{2}\theta^{ij}A_i \star_2 (\partial_j A_\mu + F_{j\mu}) \quad \text{with } f(x)\star_2 g(x) = \frac{\sin(\partial_1\theta\partial_2/2)}{\partial_1\theta\partial_2/2} f(x_1)g(x_2)\bigg|_{x_i=x}
$$

which have expansion in θ

$$
\begin{aligned}
A^{(1,\infty)}_\mu &= A^{(1,1)}_\mu + A^{(1,3)}_\mu + \cdots\,,\\
A^{(1,\infty)}_\mu &= \frac{1}{2}\left[-A\theta(\partial A_\mu + F_{\cdot\mu}) + \frac{1}{24}\,\partial\partial A\,\theta\theta\theta\,\partial\partial(\partial A_\mu + F_{\cdot\mu}) + \cdots\right].
\end{aligned}
$$

We want to use this to extend the previous recursive relation, holding for the first diagonal, to a line by line relation. The basic reason to try to sum by lines (see figure 1), is that all terms of the first line depend on $\partial A_\mu + F_{\cdot\mu}$, while terms of order (n,n) are expected to depend on $\partial A_\mu + nF_{.\mu}$, as can be seen on (3.4). We have calculated a solution at order (2,4), the second term on the second line, and indeed it happens to depend on $\partial A_\mu + 2F_{.\mu}$.

Then, one may wonder about the extended recursive rules. Recall that the expression at order 2 was the same as the solution for order 1, with

- the starting A replaced by $A^{(1,1)}$,
- $(\partial A_\mu + F.\mu)/2$ changed to $(\partial A_\mu + 2F.\mu)/3$.

If we do the same for all the line, we get

$$\begin{aligned}
\lambda^{(1,\infty)} &= \frac{1}{2}\left[-A\theta\partial\lambda + \frac{1}{24}\partial\partial A\,\theta\theta\theta\,\partial\partial\partial\lambda + \cdots\right], \\
\lambda^{(2,\infty)} &= \frac{1}{3}\left[-A^{(1,\infty)}\theta\partial\lambda + \frac{1}{24}\,\partial\partial A^{(1,\infty)}\,\theta\theta\theta\,\partial\partial\partial\lambda + \cdots\right] \\
&= \underbrace{-\frac{1}{3}A^{(1,1)}\theta\partial\lambda}_{\lambda^{(2,2)}}\ \underbrace{-\frac{1}{3}A^{(1,3)}\theta\partial\lambda + \frac{1}{72}\partial\partial A^{(1,1)}\,\theta\theta\theta\,\partial\partial\partial\lambda}_{\lambda^{(2,4)}} + \cdots .
\end{aligned}$$

And the same for $A_\mu^{(2,\infty)}$, replacing $\partial\lambda$ by $(\partial A_\mu + 2F_{.\mu})$. Do not forget that the $A^{(1,m)}$ have to act as operators (the rest of the term is “glued” inside the action of their derivatives). This prescription leads to ambiguities,[2] up to which this proposal for order (2,4) reproduces an actual solution (out of a dozen, all but one terms are reproduced unambiguously).

The same should also hold for the field strength. Generalizing the formulas (3.5) would give $F_{\mu\nu}^{(2,\infty)}$ from $F_{\mu\nu}^{(1,\infty)}$, with

$$F_{\mu\nu}^{(1,\infty)} = \begin{bmatrix} -A\theta\partial F_{\mu\nu} + \frac{1}{24}\,\partial\partial A\,\theta\theta\theta\,\partial\partial\partial F_{\mu\nu} + \cdots \\ -F_{\mu .}\theta F_{.\nu} + \frac{1}{24}\,\partial\partial F_{\mu .}\,\theta\theta\theta\,\partial\partial F_{.\nu} + \cdots \end{bmatrix} .$$

3.3 Pure gauge case

We consider here the case when the gauge potential A is a pure gauge. For an abelian theory, this can be done with $A_\rho = \partial_\rho\alpha$. As expected, the field strength vanishes, and the gauge transformation of α should be $\delta\alpha = \lambda$.

The objective here is not to find the solutions: it is the orbit of the non-commutative pure gauge. Indeed, we expect from the map that zero is mapped to zero, and this statement can only hold for the whole orbits.

What we are looking for is an expression that, in some sense, is nicer than the others. The solutions emphasized up to now will indeed greatly simplify in the pure gauge case. That will allow two different improvements, which are still out of reach for a general potential: first, the solutions have a natural non-abelian generalisation, and second, they will sum up to a non-perturbative form (as we will see in 4.1). This suggests that the same kind of expressions could hold for general potentials, once they are parametrized by elements of the gauge group.

[2] We shall try to comment on this here. In the term

$$\partial\partial A^{(1,1)}\,\theta\theta\theta\,\partial\partial\partial\lambda \propto (A\theta\,\overrightarrow{\partial})A\,(\overleftarrow{\partial\partial}\,\theta\theta\theta\,\overrightarrow{\partial\partial\partial}\lambda)\,,$$

$A^{(1,1)}$ should act as an operator on the right, while $\overleftarrow{\partial\partial}$ is acting on the left. Note that this is the first instance where such a situation occurs. One will keep meeting similar ambiguities at higher orders.

3.3.1 The solution for A and λ

We can rewrite the solution at order $(1,\infty)$

$$A_\mu^{(1,\infty)} = -\frac{1}{2}\partial\alpha\theta\partial A_\mu + \frac{1}{48}\partial\partial\partial\alpha\,\theta\theta\theta\,\partial\partial\partial A_\mu + \cdots$$

and recognize the development of a shorter formula (the same holds for $\lambda^{(1,\infty)}$)

$$\begin{aligned} A_\mu^{(1,\infty)} &= \frac{i}{2}[\alpha \overset{\star}{,} A_\mu]\,, \\ \lambda^{(1,\infty)} &= \frac{i}{2}[\alpha \overset{\star}{,} \lambda]\,. \end{aligned}$$

The Seiberg-Witten equation, written at each order in A, gives[3]

$$\begin{aligned} \text{Order } A: &\quad \delta A_\mu^{(1)} - \partial_\mu\lambda^{(1)} = i[\lambda \overset{\star}{,} A_\mu]\,, \\ \text{Order } A^2: &\quad \delta A_\mu^{(2)} - \partial_\mu\lambda^{(2)} = i\left[\lambda \overset{\star}{,} A_\mu^{(1)}\right] + i\left[\lambda^{(1)} \overset{\star}{,} A_\mu\right], \\ \text{Order } A^n: &\quad \delta A_\mu^{(n)} - \partial_\mu\lambda^{(n)} = \sum_{p=0}^{n-1} i\left[\lambda^{(p)} \overset{\star}{,} A_\mu^{(n-1-p)}\right]. \end{aligned}$$

Now we can check directly that the expression is indeed solution at order A

$$\begin{aligned} \delta A_\mu^{(1)} - \partial_\mu\lambda^{(1)} &= \frac{1}{2}i[\lambda \overset{\star}{,} A_\mu] + \frac{1}{2}i[\alpha \overset{\star}{,} \partial_\mu\lambda] - \frac{1}{2}i[\partial_\mu\alpha \overset{\star}{,} \lambda] - \frac{1}{2}i[\alpha \overset{\star}{,} \partial_\mu\lambda] \\ &= i[\lambda \overset{\star}{,} A_\mu] \quad \equiv \quad i[\lambda^{(0)} \overset{\star}{,} A_\mu^{(0)}]. \end{aligned}$$

This formula is still valid to all orders, as we shall now prove recursively. Suppose we have the solution for $k \leq n$

$$\begin{aligned} A_\mu^{(k)} &= \frac{i}{k+1}\left[\alpha \overset{\star}{,} A_\mu^{(k-1)}\right], \\ \lambda^{(k)} &= \frac{i}{k+1}\left[\alpha \overset{\star}{,} \lambda^{(k-1)}\right] \end{aligned}$$

and define order $(n+1)$ by the same recursive formula. Then

$$\delta A_\mu^{(n+1)} - \partial_\mu\lambda^{(n+1)} = \frac{i}{n+2}\left[\lambda \overset{\star}{,} A_\mu^{(n)}\right] + \frac{i}{n+2}\left[\lambda^{(n)} \overset{\star}{,} A_\mu\right] + \frac{i}{n+2}\left[\alpha \overset{\star}{,} \delta A_\mu^{(n)} - \partial_\mu\lambda^{(n)}\right]$$

and with the help of the Jacobi identity

$$\begin{aligned} \left[\alpha \overset{\star}{,} \delta A_\mu^{(n)} - \partial_\mu\lambda^{(n)}\right] &= \left[\alpha \overset{\star}{,} \sum_{p=0}^{n-1} i\left[\lambda^{(p)} \overset{\star}{,} A_\mu^{(n-1-p)}\right]\right] \\ &= \sum_{p=0}^{n-1}\left(\left[\lambda^{(p)} \overset{\star}{,} i\left[\alpha \overset{\star}{,} A_\mu^{(n-1-p)}\right]\right] - \left[A_\mu^{(n-1-p)} \overset{\star}{,} i\left[\alpha \overset{\star}{,} \lambda^{(p)}\right]\right]\right) \\ &= \sum_{p=0}^{n-1}\left((n+1-p)\left[\lambda^{(p)} \overset{\star}{,} A_\mu^{(n-p)}\right] - (p+2)\left[A_\mu^{(n-1-p)} \overset{\star}{,} \lambda^{(p+1)}\right]\right) \end{aligned}$$

[3]From now on, we drop the ∞ since no confusion can arise: $A_\mu^{(n)}$ means $A_\mu^{(n,\infty)}$.

JHEP06(2002)016

which is exactly what is needed to finally give

$$\delta A_\mu^{(n+1)} - \partial_\mu \lambda^{(n+1)} = \sum_{p=0}^{n} i \left[\lambda^{(p)} \overset{\star}{,} A_\mu^{(n-p)} \right].$$

So we arrive at a solution to all orders for $\hat{A}_\mu$ and $\hat{\lambda}$, in the form

$$\begin{aligned} A_\mu^{(n+1)} &= \frac{i}{n+2} \left[\alpha \overset{\star}{,} A_\mu^{(n)} \right], \\ \lambda^{(n+1)} &= \frac{i}{n+2} [\alpha \overset{\star}{,} \lambda^{(n)}], \end{aligned} \tag{3.6}$$

$A_\mu^{(0)}$ and $\lambda^{(0)}$ being A_μ and λ.

3.3.2 The field strength

Here we compute the field strength corresponding to the solution (3.6) for the non-commutative connection, and show recursively that it indeed vanishes. $\hat{F}_{\mu\nu}$ can be developed in powers of A, giving

$$\begin{aligned} F_{\mu\nu}^{(0)} &\equiv F_{\mu\nu} = 0\,, \\ F_{\mu\nu}^{(1)} &\equiv \partial_\mu A_\nu^{(1)} - \partial_\nu A_\mu^{(1)} - i[A_\mu \overset{\star}{,} A_\nu] \\ &= \frac{i}{2}[A_\mu \overset{\star}{,} A_\nu] + \frac{i}{2}[\alpha \overset{\star}{,} \partial_\mu A_\nu] - \frac{i}{2}[A_\nu \overset{\star}{,} A_\mu] - \frac{i}{2}[\alpha \overset{\star}{,} \partial_\nu A_\mu] - i[A_\mu \overset{\star}{,} A_\nu] \\ &= \frac{i}{2}[\alpha \overset{\star}{,} F_{\mu\nu}] = 0\,, \\ F_{\mu\nu}^{(n+1)} &\equiv \partial_\mu A_\nu^{(n+1)} - \partial_\nu A_\mu^{(n+1)} - \sum_{k=0}^{n} i \left[A_\mu^{(k)} \overset{\star}{,} A_\nu^{(n-k)} \right] \\ &= \frac{i}{n+2} \left[A_\mu \overset{\star}{,} A_\nu^{(n)} \right] - \frac{i}{n+2} \left[A_\nu \overset{\star}{,} A_\mu^{(n)} \right] + \frac{i}{n+2} \left[\alpha \overset{\star}{,} \partial_\mu A_\nu^{(n)} - \partial_\nu A_\mu^{(n)} \right] - \\ &\quad - \sum_{k=0}^{n} i \left[A_\mu^{(k)} \overset{\star}{,} A_\nu^{(n-k)} \right]. \end{aligned}$$

Using the definition of $F_{\mu\nu}^{(n)}$

$$\partial_\mu A_\nu^{(n)} - \partial_\nu A_\mu^{(n)} = F_{\mu\nu}^{(n)} + \sum_{k=0}^{n-1} i \left[A_\mu^{(k)} \overset{\star}{,} A_\nu^{(n-1-k)} \right],$$

the Jacobi identity

$$\begin{aligned} \left[\alpha \overset{\star}{,} i \left[A_\mu^{(k)} \overset{\star}{,} A_\nu^{(n-1-k)} \right] \right] &= \left[A_\mu^{(k)} \overset{\star}{,} i \left[\alpha \overset{\star}{,} A_\nu^{(n-1-k)} \right] \right] - \left[A_\nu^{(n-1-k)} \overset{\star}{,} i \left[\alpha \overset{\star}{,} A_\mu^{(k)} \right] \right] \\ &= (n-k+1) \left[A_\mu^{(k)} \overset{\star}{,} A_\nu^{(n-k)} \right] + (k+2) \left[A_\mu^{(k+1)} \overset{\star}{,} A_\nu^{(n-1-k)} \right], \end{aligned}$$

and replacing in $F_{\mu\nu}^{(n+1)}$, most of the terms cancel, leaving

$$F_{\mu\nu}^{(n+1)} = \frac{i}{n+2} \left[\alpha \overset{\star}{,} F_{\mu\nu}^{(n)} \right] = 0\,. \tag{3.7}$$

4. Non-abelian theory

4.1 The pure gauge

To go on with the pure gauge in the non-abelian theory, we have to find some parametrisation compatible with the gauge transformation

$$\delta A_\mu = \partial_\mu \lambda + i[\lambda ; A_\mu] \,.$$

Indeed, the crucial point in the abelian theory was the existence of α, such that $A_\mu = \partial_\mu \alpha$ and $\lambda = \delta\alpha$. And we already know a generalization of this, allowing the non-abelian gauge transformation:

$$\begin{aligned} A_\mu &= \partial_\mu \alpha + \frac{i}{2}\,[\alpha ; \partial_\mu \alpha] + \frac{i}{6}\,[\alpha ; i[\alpha ; \partial_\mu \alpha]] + \cdots \,, \\ \lambda &= \delta\alpha + \frac{i}{2}\,[\alpha ; \delta\alpha] + \frac{i}{6}\,[\alpha ; i[\alpha ; \delta\alpha]] + \cdots \,. \end{aligned} \tag{4.1}$$

These are exactly the formulas (3.6) for $\hat{A}_\mu$ and $\hat{\lambda}$, the commutator being now a gauge commutator instead of a star commutator. Applying the results, without the hats and the stars, we see that the gauge transformation of a non-abelian A_μ is the expected one, and that the field strength vanishes. We can also point out that all gauge commutators vanish in the abelian case, and that A_μ and λ then reduce to the shorter expected formulas.

On the non-commutative side, we still have the expressions

$$\begin{aligned} \hat{A}_\mu &= \partial_\mu \alpha + \frac{i}{2}\,[\alpha \overset{\star}{;} \partial_\mu \alpha] + \frac{i}{6}\,[\alpha \overset{\star}{;} i[\alpha \overset{\star}{;} \partial_\mu \alpha]] + \cdots \,, \\ \hat{\lambda} &= \delta\alpha + \frac{i}{2}\,[\alpha \overset{\star}{;} \delta\alpha] + \frac{i}{6}\,[\alpha \overset{\star}{;} i[\alpha \overset{\star}{;} \delta\alpha]] + \cdots \,, \end{aligned} \tag{4.2}$$

where the commutators take care at the same time of the noncommutativity and of the non-abelian gauge group. The relation between the above A_μ and $\hat{A}_\mu$ is precisely the SW map.

In particular, we get A_μ and λ for $\theta = 0$, as expected. So that the expressions of the non-commutative variables in terms of the commutative ones, order by order in θ, start like

$$\begin{aligned} \hat{A}_\mu &= A_\mu + A_\mu^{(1)} + \cdots \,, \\ \hat{\lambda} &= \lambda + \lambda^{(1)} + \cdots \,. \end{aligned}$$

To understand why such a parametrisation exists, and what it means, let us see that we can go beyond this perturbative development. If A is a pure gauge, then it can be written in terms of an element $g = e^{-i\alpha}$ of the gauge group

$$A = i g^{-1} dg \,.$$

Developed in powers of α, this expression is nothing but (4.1). The non-perbutative formula for λ is $\lambda = i g^{-1} \delta g$ (which is nothing but the gauge transformation of g). Of course, it can be checked directly that F is zero or that the right gauge transformation holds.

On the non-commutative side, the formulas are the same. We use the corresponding element $\hat{g} = (\star e)^{-i\alpha} = 1 - i\alpha - (\alpha \star \alpha/2) + \cdots$, and replace ordinary multiplications by star products

$$\begin{aligned}\hat{A} &= i\hat{g}^{-1} \star d\hat{g} = d\alpha + \frac{i}{2}[\alpha \overset{\star}{;} d\alpha] + \frac{i}{6}[\alpha \overset{\star}{;} i[\alpha \overset{\star}{;} d\alpha]] + \cdots , \\ \hat{\lambda} &= i\hat{g}^{-1} \star \delta\hat{g} = \delta\alpha + \frac{i}{2}[\alpha \overset{\star}{;} \delta\alpha] + \frac{i}{6}[\alpha \overset{\star}{;} i[\alpha \overset{\star}{;} \delta\alpha]] + \cdots . \end{aligned} \tag{4.3}$$

4.2 The general case

Is it possible to generalize the recursive picture from abelian theory to non-abelian one? To do this, we first have to find a "nice" solution at order θ^2, which means that it should at least reduce to the abelian solution that we found in 3.1.2. But with the appearance of gauge commutators, there is a lot of freedom in generalizing, and we face some new problems:

- one term out of two in figure 1 was a vanishing commutator, now it does not vanish,
- the derivative could be replaced by the covariant derivative or stay as an ordinary derivative,
- the order in A (the lines) is not well defined, now that the field strength, the gauge transformation and the covariant derivative have a quadratic part.

The last point is the reason why we have to calculate the columns first (see figure 1), and sort the terms thereafter, identifying which term belongs to which line.

Let us start with order θ. The Seiberg-Witten equation is

$$\delta A_\mu^{(\infty,1)} - i\left[\lambda; A_\mu^{(\infty,1)}\right] - \partial_\mu \lambda^{(\infty,1)} + i\left[A_\mu; \lambda^{(\infty,1)}\right] = -\frac{1}{2}\left\{\partial\lambda \overset{\theta}{;} \partial A_\mu\right\}.$$

We can point out that here δ is non abelian, but its non-abelian part is partly removed by the second term. The two other terms build up the covariant derivative of $\lambda^{(\infty,1)}$. It has the solution

$$\begin{aligned}\lambda^{(\infty,1)} &= -\frac{1}{4}\{A \overset{\theta}{;} \partial\lambda\}, \\ A_\mu^{(\infty,1)} &= -\frac{1}{4}\{A \overset{\theta}{;} \partial A_\mu + F_{\cdot\mu}\}, \\ F_{\mu\nu}^{(\infty,1)} &= -\frac{1}{2}\{F_{\mu\cdot} \overset{\theta}{;} F_{\cdot\nu}\} - \frac{1}{2}\left\{A \overset{\theta}{;} \frac{\partial + D}{2} F_{\mu\nu}\right\}, \end{aligned} \tag{4.4}$$

where D is the gauge covariant derivative: $D = \partial - i[A; \cdot]$.

As already emphasized, in the abelian case all these terms were of order A (meaning $A\lambda$ or AA_μ). Here they go to order A^3. The question is whether they all belong to the first "line" anyway, higher orders in A being hidden in D and F, or not.

At order θ^2, the equation is

$$\begin{aligned}&\delta A_\mu^{(\infty,2)} - i\left[\lambda; A_\mu^{(\infty,2)}\right] - \partial_\mu \lambda^{(\infty,2)} + i\left[A_\mu; \lambda^{(\infty,2)}\right] = \\ &= i\left[\lambda^{(\infty,1)}; A_\mu^{(\infty,1)}\right] - \frac{1}{2}\left\{\partial\lambda^{(\infty,1)} \overset{\theta}{;} \partial A_\mu\right\} - \frac{1}{2}\left\{\partial\lambda \overset{\theta}{;} \partial A_\mu^{(\infty,1)}\right\} - \frac{i}{8}\left[\partial\partial\lambda \overset{\theta\theta}{;} \partial\partial A_\mu\right]\end{aligned}$$

for which solutions are known [19, 23, 24]. We have found another one, getting closer to the more or less expected form, but only fitting it partially. If we try to guess a recursive formula, we again face the problem of identifying order (1,1) and (2,2) among their respective columns. For example, the conjectured generalization for $F_{\mu\nu}^{(2,2)}$ could be

$$F_{\mu\nu}^{(2,2)} \overset{??}{=} -\frac{1}{2}\left\{F_{\mu\cdot}^{(1,1)} \overset{\theta}{,} F_{\cdot\nu}\right\} - \frac{1}{2}\left\{A^{(1,1)} \overset{\theta}{,} \frac{\partial + 2D}{3} F_{\mu\nu}\right\}. \tag{4.5}$$

Paradoxically enough, the first part, which was harder to check in the abelian case, gives exactly the right result, while there are still problems with the second part.

Acknowledgments

I would like to thank A. Armoni and R. Minasian for useful discussions and advice.

References

[1] M.R. Douglas and N.A. Nekrasov, *Noncommutative field theory*, *Rev. Mod. Phys.* **73** (2002) 977 [hep-th/0106048].

[2] R.J. Szabo, *Quantum field theory on noncommutative spaces*, hep-th/0109162.

[3] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, *J. High Energy Phys.* **09** (1999) 032 [hep-th/9908142].

[4] K. Saraikin, *Comments on the morita equivalence*, *J. Exp. Theor. Phys.* **91** (2000) 653 [hep-th/0005138].

[5] B. Pioline and A. Schwarz, *Morita equivalence and T-duality (or B versus θ)*, *J. High Energy Phys.* **08** (1999) 021 [hep-th/9908019].

[6] A.A. Tseytlin, *On non-abelian generalisation of the Born-Infeld action in string theory*, *Nucl. Phys.* **B 501** (1997) 41 [hep-th/9701125].

[7] L. Cornalba, *On the general structure of the non-abelian Born-Infeld action*, hep-th/0006018.

[8] L. Cornalba, *Some computations with Seiberg-Witten invariant actions*, *Mod. Phys. Lett.* **A 16** (2001) 227 [hep-th/0101015].

[9] N. Wyllard, *Derivative corrections to the D-brane Born-Infeld action: non-geodesic embeddings and the seiberg-Witten map*, *J. High Energy Phys.* **08** (2001) 027 [hep-th/0107185].

[10] H. Liu, *$*$-trek II: $*_n$ operations, open Wilson lines and the Seiberg-Witten map*, *Nucl. Phys.* **B 614** (2001) 305 [hep-th/0011125].

[11] H. Liu and J. Michelson, *Ramond-Ramond couplings of noncommutative D-branes*, *Phys. Lett.* **B 518** (2001) 143 [hep-th/0104139].

[12] Y. Okawa and H. Ooguri, *An exact solution to Seiberg-Witten equation of noncommutative gauge theory*, *Phys. Rev.* **D 64** (2001) 046009 [hep-th/0104036].

[13] S.R. Das and S.P. Trivedi, *Supergravity couplings to noncommutative branes, open Wilson lines and generalized star products*, *J. High Energy Phys.* **02** (2001) 046 [hep-th/0011131].

[14] S. Mukhi and N.V. Suryanarayana, *Gauge-invariant couplings of noncommutative branes to Ramond-Ramond backgrounds*, *J. High Energy Phys.* **05** (2001) 023 [hep-th/0104045].

[15] S. Mukhi and N.V. Suryanarayana, *Gauge-invariant couplings of noncommutative branes to Ramond-Ramond backgrounds*, *J. High Energy Phys.* **05** (2001) 023 [hep-th/0104045].

[16] T. Mehen and M.B. Wise, *Generalized $*$-products, Wilson lines and the solution of the Seiberg-Witten equations*, *J. High Energy Phys.* **12** (2000) 008 [hep-th/0010204].

[17] S.S. Pal, *Derivative corrections to Dirac-Born-Infeld and Chern-Simon actions from non-commutativity*, *Int. J. Mod. Phys.* **A 17** (2002) 1253 [hep-th/0108104].

[18] T. Asakawa and I. Kishimoto, *Comments on gauge equivalence in noncommutative geometry*, *J. High Energy Phys.* **11** (1999) 024 [hep-th/9909139].

[19] B. Jurčo, L. Möller, S. Schraml, P. Schupp and J. Wess, *Construction of non-abelian gauge theories on noncommutative spaces*, *Eur. Phys. J.* **C 21** (2001) 383 [hep-th/0104153].

[20] P. Schupp, *Non-abelian gauge theory on noncommutative spaces*, hep-th/0111038.

[21] B. Jurčo, P. Schupp and J. Wess, *Nonabelian noncommutative gauge fields and Seiberg-Witten map*, *Mod. Phys. Lett.* **A 16** (2001) 343 [hep-th/0012225].

[22] J. Madore, S. Schraml, P. Schupp and J. Wess, *Gauge theory on noncommutative spaces*, *Eur. Phys. J.* **C 16** (2000) 161 [hep-th/0001203].

[23] S. Goto and H. Hata, *Noncommutative monopole at the second order in theta*, *Phys. Rev.* **D 62** (2000) 085022 [hep-th/0005101].

[24] D. Brace, B.L. Cerchiai, A.F. Pasqua, U. Varadarajan and B. Zumino, *A cohomological approach to the non-abelian Seiberg-Witten map*, *J. High Energy Phys.* **06** (2001) 047 [hep-th/0105192].

JHEP06(2002)016